The Invariant Representations of a Quadric Cone and a Twisted Cubic

Y.H. Wu and Z.Y. Hu

Abstract—Up to now, the shortest invariant representation of a quadric has 138 summands and there has been no invariant representation of a twisted cubic in 3D projective space, which limit to some extent the applications of invariants in 3D space. In this paper, we give a very short invariant representation of a quadric cone, a special quadric, which has only two summands similar to the invariant representation of a planar conic, and give a short invariant representation of a twisted cubic. Then, a completely linear algorithm for generating the parametric equations of a twisted cubic is provided also. Finally, we exemplify some applications of our proposed invariant representations in the fields of computer vision and automated geometric theorem proving.

Index Terms—Automated theorem proving, computer vision, invariant representation, quadric cone, twisted cubic.

1 Introduction

Invariant representations, which essentially are coordinate-free algebraic expressions, have been frequently used in the fields of computer vision [8], [9], [11], [12], [13], automated geometric theorem proving [6], [7], [19], etc. Generally speaking, the invariant representations in 2D geometric space have attracted more attention than those in 3D geometric space. One of the underlying reasons is that the invariant representations of pure 3D geometric elements, such as quadrics, cubic curves, and cubic surfaces, have not been fully developed in the literature.

The invariant condition for 10 points in 3D projective space to lie on a quadric surface was first clarified by Turnbull and Young [17] in 1926. White [18] rewrote the result into a sum of 138 monomials in the determinants (or brackets) in 1989. But, the disadvantages of both the representations are too long to be manageable in real applications.

For a twisted cubic in 3D projective space, we know that it can be uniquely determined by six generic points and its representation usually is a system of parametric equations or quadric equations under some coordinate system [15]. To our knowledge, its invariant representation has not been available. Worse still, even the coordinates of six points under some coordinate system are known, its parametric equations cannot be obtained directly (there is an algorithm to present the parametric equations in [8], [9]), and its quadric equations cannot be obtained easily either because usually there is a unique quadric passing through nine points.

The main points of this manuscript are:

1. We give an invariant representation of a quadric cone and an invariant representation of a twisted cubic, which are all very short and each has only two summands, derived from the geometric viewpoint rather than from the algebraic viewpoint as by Turnbull and Young [17] and White [18].
2. We present an algorithm to generate the parametric equations of a twisted cubic, which is completely linear, and not any factorization is involved compared with the method in [8], [9].
3. We illustrate by examples the applicability of our invariant representations in computer vision and automated geometric theorem proving.

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The paper is organized as follows: Section 2 gives some preliminaries. In Section 3, the invariant representation of a quadric cone, the invariant representation of a twisted cubic, and a linear algorithm to generate the parametric equations of a twisted cubic are elaborated. Then, the potential applications of these results in computer vision and automated geometric theorem proving are exemplified in Section 4 and Section 5, respectively. In Section 6 are some concluding remarks.

2 Preliminaries

In the paper, a bold number denotes a homogeneous vector, a bold capital letter denotes either a homogeneous vector or a matrix, a bracket "[ ]" denotes a determinant of vectors in it, and the symbol "$\approx$ " denotes the equality up to a scale.

Definition 1. *The locus of points* $\mathbf{X}$ *in 3D projective space satisfying the equation:*

$$\mathbf{X}^{\tau}\mathbf{M}\mathbf{X} = 0 \tag{1}$$

is a quadric, where $\mathbf{M}$ *is a* 4×4 *symmetric matrix. More specifically, if there exists a unique point* $\mathbf{V}$ *on the locus such that, for any point* $\mathbf{P}$ *on the locus, the whole line* $\mathbf{VP}$ *is on the locus, then the locus is a proper quadric cone, and* $\mathbf{V}$ *is called its vertex. At the time,* $rank(\mathbf{M}) = 3$*, the kernel of* $\mathbf{M}$*, i.e., the right null vector of* $\mathbf{M}$*, is the vertex* $\mathbf{V}$*.*

When $rank(\mathbf{M}) = 4$, the locus of (1) is a proper (nondegenerate) quadric and, when $rank(\mathbf{M}) = 2$ or $rank(\mathbf{M}) = 1$, it is a pair of planes or a repeated plane.

Definition 2. *The locus of points* $\mathbf{X} = (X\ Y\ Z\ T)^{\tau}$ *in 3D projective space satisfying the parametric equation:*

$$(X\ Y\ Z\ T)^{\tau} \approx \mathbf{M}\left(\theta^3\ \theta^2\ \theta\ 1\right)^{\tau} \tag{2}$$

is a twisted cubic, where $\mathbf{M}$ *is a* 4×4 *matrix and* θ *is the parameter. When the rank of* $\mathbf{M}$ *is 4, then the locus of (2) is a proper twisted cubic.*

The following Lemma 1 and Lemma 2 shown in [15, p. 301 and 314] are needed in this paper.

Lemma 1. *If* $\mathbf{1},\ \mathbf{2},\ \mathbf{3},\ \mathbf{4}$ *are four fixed points of a twisted cubic* C *and* L *is a chord of* C*, then the cross ratio* $L(\mathbf{1},\ \mathbf{2};\ \mathbf{3},\ \mathbf{4})$ *of four axial planes with axis* L *and passing through* $\mathbf{1},\ \mathbf{2},\ \mathbf{3},\ \mathbf{4}$*, respectively, is independent of the choice of* L*.*

Lemma 2. *There is a unique twisted cubic that passes through six general space points.*

Let $\mathbf{H}$ *be a nondegenerate matrix with determinant* 1*, a polynomial or rational polynomial* $I(\mathbf{A}_1,\ldots,\mathbf{A}_n)$ *with respect to vectors* $\mathbf{A}_1,\ldots,\mathbf{A}_n$ *is an invariant if:*

$$I(\mathbf{A}_1,\ldots,\mathbf{A}_n) = I(\mathbf{H}\mathbf{A}_1,\ldots,\mathbf{H}\mathbf{A}_n).$$

In this paper, an invariant representation is an equation with respect to some invariant $I(\mathbf{A}_1,\ldots,\mathbf{A}_n)$, i.e.,

$$I(\mathbf{A}_1,\ldots,\mathbf{A}_n) = 0.$$

3 The Invariant Representations of a Quadric Cone and a Twisted Cubic

The main theorems of this paper are presented in this section.

Theorem 1.

1. *Given six points* $\mathbf{1},\ \mathbf{2},\ \mathbf{3},\ \mathbf{4},\ \mathbf{5},\ \mathbf{6}$ *with no three collinear and no four coplanar, there is a unique proper quadric cone through points* $\mathbf{2},\ \mathbf{3},\ \mathbf{4},\ \mathbf{5},\ \mathbf{6}$ *and with vertex* $\mathbf{1}$*.*

2. *Any point* $\mathbf{X}$ *on the above proper quadric cone in Step 1 satisfies*

$$[\mathbf{124X}][\mathbf{135X}] - \frac{[\mathbf{1246}][\mathbf{1356}]}{[\mathbf{1236}][\mathbf{1456}]}[\mathbf{123X}][\mathbf{145X}] = 0. \quad (3)$$

The representation of the equation is not unique as a result that the equation after a permutation of $\mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}$ *is also a representation of the cone.*

Proof.

1. By Definition 1, $\mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}$ give five linear constraints on $\mathbf{M}$, and vertex $\mathbf{1}$ gives four linear constraints on $\mathbf{M}$ (i.e., $\mathbf{M}\,\mathbf{1} = 0$). If the nine constraints are independent, then $\mathbf{M}$ (up to a scale) can be uniquely determined by $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}$ and the quadric cone through points $\mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}$ with vertex $\mathbf{1}$ is unique. Suppose that the nine constraints were not independent, then there should exist at least two quadric cones through points $\mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}$ and with vertex $\mathbf{1}$. By a perspective projection of points $\mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}$ on a plane not going through perspective center $\mathbf{1}$, the images of $\mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}$ will be distinct and no three of them are collinear, since no three of $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}$ are collinear and no four of them are coplanar. Then, at least two distinct conics will be formed, all going through the five images of $\mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}$. However, since there is a unique conic going through five generic points, the assumption of the dependence of the nine constraints must be untrue. In other words, there is only a unique quadric cone going through $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}$ with vertex $\mathbf{1}$.
2. It is clear that (3) is a quadric equation with respect to $\mathbf{X}$, by (1) the locus of (3) is a quadric. Let $\mathbf{X}$ be $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}$, (3) is always true, thus the quadric determined by (3) is passing through $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}$. Assume that $\mathbf{A}$ is any point satisfying (3) other than $\mathbf{1}$, i.e.,

$$[\mathbf{124A}][\mathbf{135A}] - \frac{[\mathbf{1246}][\mathbf{1356}]}{[\mathbf{1236}][\mathbf{1456}]}[\mathbf{123A}][\mathbf{145A}] = 0, \quad (4)$$

any point $\mathbf{B}$ on the line $\mathbf{1A}$ other than $\mathbf{A}$ can be represented by:

$$\mathbf{B} = \mathbf{1} + \lambda\,\mathbf{A},$$

where λ is a scalar. Substitute $\mathbf{B} = \mathbf{1} + \lambda\,\mathbf{A}$ for $\mathbf{X}$ in (3), by (4), we have:

$$\begin{aligned}&[\mathbf{124B}][\mathbf{135B}] - \frac{[\mathbf{1246}][\mathbf{1356}]}{[\mathbf{1236}][\mathbf{1456}]}[\mathbf{123B}][\mathbf{145B}]\\ &= \lambda^2\left([\mathbf{124A}][\mathbf{135A}] - \frac{[\mathbf{1246}][\mathbf{1356}]}{[\mathbf{1236}][\mathbf{1456}]}[\mathbf{123A}][\mathbf{145A}]\right)\\ &= 0.\end{aligned}$$

Thus, we know that, if $\mathbf{A}$ is a point on the quadric determined by (3), then the whole line $\mathbf{1A}$ is on it. Furthermore, the locus of (3) cannot be degenerate planes because no three of $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}$ are collinear and no four of them are coplanar. Therefore, (3) can only be the equation of the proper quadric cone through points $\mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}$ with vertex $\mathbf{1}$.

The above proof is independent of the order of $\mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}$, so (3) after a permutation of $\mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}$ is also a representation of the cone. □

Theorem 2. *Any point* $\mathbf{X}$ *is on the proper twisted cubic passing through six points* $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}$ *with no three collinear and no four coplanar if and only if:*

1.

$$\begin{cases}[\mathbf{124X}][\mathbf{135X}] - \dfrac{[\mathbf{1246}]\,[\mathbf{1356}]}{[\mathbf{1236}]\,[\mathbf{1456}]}[\mathbf{123X}][\mathbf{145X}] = 0,\\[2mm] [\mathbf{124X}][\mathbf{235X}] - \dfrac{[\mathbf{1246}]\,[\mathbf{2356}]}{[\mathbf{1236}]\,[\mathbf{2456}]}[\mathbf{123X}][\mathbf{245X}] = 0,\end{cases} \quad (5)$$

2. $\mathbf{X}$ *is not on the line* $\mathbf{12}$ *except for* $\mathbf{1}, \mathbf{2}$.

The above representation is not unique as a result that the one after a permutation of $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}$ *is also a representation of the twisted cubic.*

Proof. "$\Rightarrow$ " " if part"

1. By Lemma 2, we know that the six points $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}$ determine a unique twisted cubic C. It is clear that $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}$ all satisfy (5). Assume that $\mathbf{X}$ is any point other than $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}$ on C, then by Lemma 1, the cross ratio of the four coaxial planes $\mathbf{162}, \mathbf{163}, \mathbf{164}, \mathbf{165}$:

$$\frac{[\mathbf{1246}][\mathbf{1356}]}{[\mathbf{1236}][\mathbf{1456}]}$$

is equal to the cross ratio of the four coaxial planes $\mathbf{1X2}, \mathbf{1X3}, \mathbf{1X4}, \mathbf{1X5}$:

$$\frac{[\mathbf{124X}][\mathbf{135X}]}{[\mathbf{123X}][\mathbf{145X}]}.$$

So, the first equation in (5) is true. Similarly, because the cross ratio of the four coaxial planes $\mathbf{261}, \mathbf{263}, \mathbf{264}, \mathbf{265}$ is equal to the cross ratio of the four coaxial planes $\mathbf{2X1}, \mathbf{2X3}, \mathbf{2X4}, \mathbf{2X5}$, we have the second equation in (5).
2. For a proper twisted cubic, since no three points on it are collinear, $\mathbf{X}$ is not on the line $\mathbf{12}$ except for at $\mathbf{1}, \mathbf{2}$.

"$\Leftarrow$ ": "only if" part

Now, we consider a point $\mathbf{X}$ other than $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}$ on the locus determined by Steps 1 and 2. Because no three of $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}$ are collinear and no four of them are coplanar, we can set up a projective coordinate system such that $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}$ have the coordinates:

$$(1\;0\;0\;0)^\tau, (0\;1\;0\;0)^\tau, (0\;0\;1\;0)^\tau, (0\;0\;0\;1)^\tau, (1\;1\;1\;1)^\tau.$$

Assume that the coordinates of $\mathbf{6}$ is $(a\;b\;c\;d)^\tau$, the coordinates of $\mathbf{X}$ is $(X\;Y\;Z\;T)^\tau$, then the Conditions 1 and 2 in the theorem are changed into:

$$\begin{cases}d(b-c)\,Y\,Z + c(d-b)\,Y\,T + b(c-d)\,Z\,T = 0,\\ d(a-c)\,X\,Z + c(d-a)\,X\,T + a(c-d)\,Z\,T = 0,\\ Z \neq 0, \quad T \neq 0.\end{cases} \quad (6)$$

We can infer that the coefficients of Y, X in (6) cannot be zero, i.e.,

$$s_1 = d(b-c)\;Z + c(d-b)\;T \neq 0,$$

$$s_2 = d(a-c)\;Z + c(d-a)\;T \neq 0,$$

otherwise $b(c-d) = 0$ or $a(c-d) = 0$ which implies that there exist four points among $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}$ to be coplanar. It follows that we have an equivalent form of (6):

$$\begin{cases}Y = \frac{b(d-c)ZT}{d(b-c)Z+c(d-b)T},\\ X = \frac{a(d-c)ZT}{d(a-c)Z+c(d-a)T}.\end{cases} \quad (7)$$

Denote $\frac{Z}{T}$ as θ, by (7), we know that $\mathbf{X}$ satisfies:

$$s_1 s_2 \begin{pmatrix} X \\ Y \\ Z \\ T \end{pmatrix} = T^3 \mathbf{H} \begin{pmatrix} \theta^3 \\ \theta^2 \\ \theta \\ 1 \end{pmatrix}, \quad (8)$$

where $\mathbf{H} = (H_{ij})$ is a 4×4 matrix with:

$$\begin{aligned} &H_{11} = H_{14} = H_{21} = H_{24} = H_{34} = H_{41} = 0, \\ &H_{12} = ad(d-c)(b-c), H_{13} = ac(c-d)(b-d), \\ &H_{22} = bd(d-c)(a-c), H_{23} = bc(c-d)(a-d), \\ &H_{31} = H_{42} = d^2(a-c)(b-c), \\ &H_{32} = H_{43} = dc(ac + ad + bc + bd - 2ab - 2dc), \\ &H_{33} = H_{44} = c^2(a-d)(b-d). \end{aligned}$$

The determinant of $\mathbf{H}$ is:

$$abc^3d^3(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)^3$$

which is nonzero because no four of $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}$ are coplanar. Thus, by Definition 2, we know that $\mathbf{X}$ is a point on a proper twisted cubic. On the other hand, it is clear that $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}$ are on the locus determined by conditions 1 and 2 in the theorem. Since six points determine a unique twisted cubic (see Lemma 2), the twisted cubic is just the one determined by $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}$.

The above proof is independent of the order of $\mathbf{1}$, $\mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}$, so Steps 1 and 2 after a permutation of $\mathbf{1}, \mathbf{2}$, $\mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}$ is also a representation of the twisted cubic. □

Following the steps in Theorem 2 proof, we can obtain a linear algorithm to generate the parametric equations of a proper twisted cubic when the coordinates of six points under some coordinate system are given.

An algorithm to generate the parametric equations of a twisted cubic

Input: The coordinates $\mathbf{A}_i,\ i = 1 \cdots 6$ of six points of the twisted cubic under some coordinate system M_0.

Output: A matrix $\mathbf{M}$ that is the 4×4 matrix in the parametric representation (2) under the coordinate system M_0.

Step 1: Let $\mathbf{B}_i,\ i = 1..5$ are, respectively, $(1\ 0\ 0\ 0)^\tau$, $(0\ 1\ 0\ 0)^\tau$, $(0\ 0\ 1\ 0)^\tau$, $(0\ 0\ 0\ 1)^\tau$, $(1\ 1\ 1\ 1)^\tau$, then solve for $\mathbf{H}_0$ from the equations $\mathbf{A}_i \approx \mathbf{H}_0 \mathbf{B}_i,\ i = 1..5$, and let $\mathbf{B}_6 = \mathbf{H}_0^{-1}\mathbf{A}_6$.

Step2: Compute the matrix $\mathbf{H}$ in (8) by substituting the coordinates $\mathbf{B}_6$ for $(a\ b\ c\ d)^\tau$.

Step 3: Let $\mathbf{M} = \mathbf{H}_0\mathbf{H}$, which is the 4×4 matrix in the parametric representation (2) under the coordinate system M_0.

4 Application in Computer Vision

The properties of a twisted cubic underlie many of the ambiguous cases which arise in reconstruction [1], [4], [10], [2], and underlie the invariants of points and lines in images and space [8], [9]. The invariant representation of a twisted cubic showed in Section 3 is free of coordinate systems, which implies that it can be conveniently used in computer vision. An example is given below.

Accurate and robust estimation of the fundamental matrix is a very important step for 3D reconstruction from 2D images. In [5], Lei et al. gave a robust RASANC-based algorithm to estimate the fundamental matrices between each pair of three images, where the involved minimal sample set contains six triples of corresponding points across the three images. The main steps of their algorithm are as follows:

1. Extract triples of corresponding points across the three images and then take minimal sample sets at random from these triples.
2. For each sample set, assuming that $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}$ are the 3D space points of the six triples of corresponding points, take the coordinates of $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}$ as the canonical projective basis, and $\mathbf{6}$ is reconstructed from the sample. Then, each projective matrix of the three images is obtained.

Let S be the set of all triples of projective matrices computed from all samples.

3. Choose the best triple of projective matrices from S based on some criterion, and optimize these projective matrices, then compute the fundamental matrices between each pair of the three images from these optimized three projective matrices.

In Step 2, when one of the three optical centers, denoted C_0, and the space points $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}$ lie on a twisted cubic, $\mathbf{6}$ may be reconstructed uniquely but the projective matrix of the image associated with C_0 cannot be determined uniquely. Then, the image sample, called ambiguous sample, needs to be discarded. Unfortunately, they did not consider this problem in their work. Now, we propose an approach to detect ambiguous samples according to our new invariant representations.

Of course, a possible way to determine whether the reconstructed projective matrix is unique or not is to directly verify the resulting matrices in the above Step 2 for each sample, however, the process is time consuming. Another possible way is to check whether the six image points lie on a conic or not (if six space points and the optical center lie on a twisted cubic, then the images of the six points lie on a conic); however, such a check is merely a necessary condition, not a sufficient condition as it risks discarding robust samples.

Here is our approach and the ambiguous samples can be detected more efficiently. First, an initial projective reconstruction is done, then choose the optical center C_0 and six 3D space points whose image points lie on a conic in the image associated with C_0, to determine whether they satisfy our representation of a twisted cubic or not. If the answer is yes, then discard the sample. Our representation is both sufficient and necessary, and is free of choice of coordinate systems. Based on such a test, the ambiguous samples generated by six generic space points (see [1]) can be discarded by the initial reconstructed coordinates.

The above idea can also be applied to the similar cases as in [3], [13], [14].

5 Application in Automated Theorem Proving

A determinant of three or four homogeneous vectors in 2D or 3D space is also called a bracket. Bracket algebra is a computational tool for invariants, and its applications touch upon not only computer vision, but also geometric reasoning.

In [6], [7], [19], a general algorithm for automated theorem proving with bracket algebra was given, and more than 150 theorems containing 2D and 3D incidence geometric theorems, 2D conic geometric theorems in projective and affine space are tested. One of the difficulties that the algorithm cannot be applied to more theorems is that the algorithm needs the geometric elements in the theorems to be represented by brackets. Since we have introduced the bracket representations of a quadric cone and a twisted cubic in Theorem 1 and Theorem 2, the scope of the algorithm's applicability can be enlarged. An example is showed below.

We refer to "$\vee$" and "$\wedge$" as "join" and "meet" operations in Grassmann-Cayley algebra [16], then $\mathbf{123} \wedge \mathbf{456}$ means the intersection line of plane $\mathbf{123}$ and plane $\mathbf{456}$, and refer to $cone(\mathbf{1}, \mathbf{23456})$ as the quadric cone through points $\mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}$ and with vertex $\mathbf{1}$.

Example (Pascal theorem [16] on a quadric cone). $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}$ are generic free points in a 3D projective space, $\mathbf{7}$ is a point on $cone(\mathbf{1}, \mathbf{23456})$, then the three intersection lines $\mathbf{135} \wedge \mathbf{126}$, $\mathbf{145} \wedge \mathbf{127}$, $\mathbf{137} \wedge \mathbf{146}$ are coplanar.

The algorithm in [6], [7], [19] is first to construct a theorem and to change the conclusion of the theorem into bracket polynomial, then to eliminate the constructed points in the conclusion according to an order.

We give the construction of the theorem:

Free points: $\mathbf{1},\ \mathbf{2},\ \mathbf{3},\ \mathbf{4},\ \mathbf{5},\ \mathbf{6}$.

Semifree points: $\mathbf{7}$ is on the $cone(\mathbf{1},\ \mathbf{23456})$, $\mathbf{8}$ is on the line of $\mathbf{135} \wedge \mathbf{126}$, $\mathbf{9}$ is on the line $\mathbf{145} \wedge \mathbf{127}$.

Conclusion: the two lines $\mathbf{137} \wedge \mathbf{146}$ and $\mathbf{89}$ are coplanar.

By applying the algorithm in [6], [7], [19], the proof procedure of the theorem is:

$$(\mathbf{137} \wedge \mathbf{146}) \vee \mathbf{89}$$

(Expand into brackets by Grassmann-Cayley algebra)

$$= [\mathbf{1347}]\,[\mathbf{1689}] - [\mathbf{1367}]\,[\mathbf{1489}]$$

(Eliminate the points $\mathbf{8}$, $\mathbf{9}$)

$$\begin{aligned} = &\ [\mathbf{1235}]\,[\mathbf{1236}]\,[\mathbf{1456}]\,[\mathbf{1347}]\,[\mathbf{1247}] \\ &+ [\mathbf{1345}]\,[\mathbf{1346}]\,[\mathbf{1256}]\,[\mathbf{1247}]\,[\mathbf{1237}] \\ &- [\mathbf{1245}]\,[\mathbf{1246}]\,[\mathbf{1356}]\,[\mathbf{1347}]\,[\mathbf{1237}] \end{aligned}$$

(Eliminate the point $\mathbf{7}$)

$$= \quad 0,$$

where the eliminations of the points $\mathbf{8}$, $\mathbf{9}$ are according to the methods in [6], [19], and the elimination of $\mathbf{7}$ is according to the equations in Theorem 1 as shown in the following:

Because $\mathbf{7}$ is on the $cone(\mathbf{1},\ \mathbf{23456})$, $\mathbf{7}$ satisfies (3) in Theorem 1:

$$\begin{aligned} &[\mathbf{1236}][\mathbf{1456}][\mathbf{1247}][\mathbf{1357}] \\ &- [\mathbf{1246}][\mathbf{1356}][\mathbf{1237}][\mathbf{1457}] = 0, \end{aligned} \tag{9}$$

substitute $[\mathbf{1357}]$ and $[\mathbf{1457}]$ with their equivalent forms changed by Grassmann-Plücker relations [16]:

$$[\mathbf{1357}] = \frac{[\mathbf{1235}]\,[\mathbf{1347}] - [\mathbf{1345}]\,[\mathbf{1237}]}{[\mathbf{1234}]},$$

$$[\mathbf{1457}] = \frac{[\mathbf{1245}]\,[\mathbf{1347}] - [\mathbf{1345}]\,[\mathbf{1247}]}{[\mathbf{1234}]}$$

into (9), and by taking the numeration of the result, we have the equation:

$$\begin{aligned} &[\mathbf{1235}]\,[\mathbf{1236}]\,[\mathbf{1456}]\,[\mathbf{1347}]\,[\mathbf{1247}] \\ &+ [\mathbf{1345}]\,[\mathbf{1346}]\,[\mathbf{1256}]\,[\mathbf{1247}]\,[\mathbf{1237}] \\ &- [\mathbf{1245}]\,[\mathbf{1246}]\,[\mathbf{1356}]\,[\mathbf{1347}]\,[\mathbf{1237}] = 0. \end{aligned}$$

6 Conclusion

We have presented an invariant representation of a quadric cone and that of a twisted cubic, and their potential applications in computer vision and automated geometric theorem proving. In addition, a new linear algorithm to generate the parametric equations of a twisted cubic is provided. As we know, invariant representations usually are very sensitive to noise when applied in computer vision, preliminary simulations show that our proposed representations are not robust either. In view of this, more efficient invariant representations (like in [11]) of a quadric cone and a twisted cubic based on our representations will be studied in the future.

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References

[1] T. Buchanan, "The Twisted Cubic and Camera Calibration," *Computer Vision, Graphics, and Image Processing,* vol. 42, no. 1, pp. 130-312, 1988.

[2] R. Hartley and A. Zisserman, *Multiple View Geometry in Computer Vision.* Cambridge Univ. Press, 2000.

[3] R. Hartley and N.Y. Dano, "Reconstruction from Six-Point Sequences," *Proc. IEEE Conf. Computer Vision and Pattern Recognition,* vol. 2, pp. 480-486, 2000.

[4] F. Kahl and K. Aström, "Ambiguous Configurations for the 1D Structure and Motion Problem," *Proc. Int'l Conf. Computer Vision,* pp. 184-189, 2001.

[5] C. Lei, H. Tsui, and Z.Y. Hu, "On the Automatic Estimation of Fundamental Matrix," *Proc. Asian Conf. Computer Vision,* pp. 664-669, 2002.

[6] H.B. Li and Y.H. Wu, "Automated Short Proof Generation for Projective Geometric Theorems with Cayley and Bracket Algebras, I. Incidence Geometry," *J. Symbolic Computation,* 2002. pending publication.

[7] H.B. Li and Y.H. Wu, "Automated Short Proof Generation for Projective Geometric Theorems with Cayley and Bracket Algebras, II. Conic Geometry," *J. Symbolic Computation,* 2002. pending publication.

[8] S. Maybank, "Relation between 3D Invariants and 2D Invariants," *Proc. IEEE Workshop Representation of Visual Scenes,* June 1995.

[9] S. Maybank, "Application of the Twisted Cubic to Model Based Vision," *Proc. Workshop Geometrical Modeling and Invariants for Computer Vision,* pp. 158-165, Apr. 1995.

[10] S. Maybank, *Theory of Reconstruction from Image Motion.* Springer-Verlag, 1993.

[11] P. Meer, R. Lenz, and S. Ramakrishna, "Efficient Invariant Representations," *Int'l J. Computer Vision,* vol. 26, no. 2, pp. 137-152, 1998.

[12] *Geometric Invariance in Computer Vision,* J. Mundy and A. Zisserman, eds. Cambridge, Mass.: The MIT Press, 1992.

[13] L. Quan, "Invariants of Six Points and Projective Reconstruction from Three Uncalibrated Images," *IEEE Trans. Pattern Analysis and Machine Intelligence,* vol. 17, no. 1, pp. 34-46, Jan. 1995.

[14] F. Schaffalitzky, A. Zisserman, R. Hartley, and P. Torr, "A Six Point Solution for Structure and Motion," *Proc. European Conf. Computer Vision,* vol. 1, pp. 632-648, 2000.

[15] J. Semple and G. Kneebone, *Algebraic Projective Geometry.* Oxford Univ. Press, 1952.

[16] B. Sturmfels, *Algorithm in Invariant Theory.* Wien, N.Y.: Springer-Verlag, 1993.

[17] H.W. Turnbull and A. Young, "The Linear Invariants of Ten Quaternary Quadrics," *Trans. Cambridge Philosophical Soc.,* vol. 23 pp. 265-301, 1926.

[18] N. White, "Implementation of the Straightening Algorithm of Classical Invariant Theory," *Invariant Theory and Tableaux, IMA Volumes in Math. and Its Applications,* D. Stanton ed., vol. 19, New York: Springer, 1989.

[19] Y.H. Wu, "Bracket Algebra, Affine Bracket Algebra and Automated Geometric Proving," PhD dissertation, Inst. of Systems Science, Chinese Academy of Sciences, Beijing 2001.